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UNIFORM AIRY-TYPE EXPANSIONS OF INTEGRALS*

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Abstract. A new method for representing the remainder and coefficients in Airy-type expansions of integrals is given. The quantities are written in terms of Cauchy-type integrals and are natural generalizations of integral representations of Taylor coefficients and remainders of analytic functions. The new approach gives a general method for extending the domain of the saddle-point parameter to unbounded domains. As a side result the conditions under which the Airy-type asymptotic expansion has a double asymptotic property become clear. An example relating to Laguerre polynomials is worked out in detail. How to apply the method to other types of uniform expansions, for example, to an expansion with Bessel functions as approximants, is explained. In this case the domain of validity can be extended to unbounded domains and the double asymptotic property can be established as well.

Key words. uniform asymptotic expansions of integrals, Airy approximation, Bessel function, Laguerre polynomial, Bessel approximation

AMS subject classifications. 41A60, 30E20, 33A40, 33A65

1. Introduction.

Many problems in mathematical physics and special functions lead to integral representations of the form

$$F(z,\alpha) = \int_C e^{zf(x,\alpha)} g(x)\,dx, \tag{1.1}$$

where C is a contour in the complex plane, z is a large parameter, and f and g are analytic functions on a neighborhood of C. In Airy-type expansions f depends on a parameter α, the *saddle-point parameter*, that describes the location of the saddle points. For a critical value of α, say, $\alpha = 0$, two saddle points coalesce with each other. With the cubic transformation $x \mapsto w$, given by

$$f(x,\alpha) = \tfrac{1}{3}w^3 - b^2 w + c \tag{1.2}$$

and suggested by Chester, Friedman, and Ursell [3], an asymptotic expansion for large values of z in terms of Airy functions can be obtained, this expansion being uniformly valid with respect to α as α ranges over a connected set containing the critical value 0 in its interior. The parameters b and c are determined explicitly from the requirement that the transformation (1.2) is analytic on a neighborhood of the two saddle points. Transformation (1.2) yields the standard form

$$\frac{1}{2\pi i}\int_{\mathcal{L}} e^{z(\frac{1}{3}w^3 - b^2 w)} h_0(w)\,dw, \tag{1.3}$$

where

$$h_0(w) = g(x(w))\frac{dx}{dw}.$$

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The phase function has two saddle points at $w = \pm b$. In the transformed integral (1.3) we call b the saddle-point parameter. The integral (1.3) has a turning-point character: the behavior changes strongly when b varies from real to imaginary values. When $b = 0$, the saddle points coalesce at $w = 0$.

The method for obtaining the Airy-type expansion, based on an integration-by-parts method, is introduced for a different class of integrals in Bleistein [1]. It is described for the Airy case in Bleistein and Handelsman [2, §9.2], in Olver [10, §§9.12, 9.13], and in Wong [19, §7.5].

The purpose of the paper is to present a new method for representing the remainder (and the coefficients) in Airy-type expansions. Two new aspects with respect to the saddle-point parameter are introduced in this way.

(i) A general method is described for extending the domain of this parameter to unbounded domains, by taking into account the singularities of the integrand function (especially the distance between the singularities and the relevant saddle point). The extension is possible since the order estimates of the remainder include information on the behavior of the remainder as the saddle-point parameter tends to infinity.

(ii) The method clearly describes the condition needed for the double asymptotic property of the expansion. That is, under certain conditions, the roles of the large parameter and the saddle-point parameter may be interchanged in describing the asymptotic phenomena. For instance, our analysis shows that the Airy-type expansion of the Laguerre polynomials given in [7] does not have the double asymptotic property, although the domain of uniform validity is indeed unbounded, as is claimed in [7].

Our method is based on a new class of rational functions with which the remainders in the expansions can be represented in a manner that is analogous to the representation of the remainder in the Taylor series of an analytic function. The rational functions do not depend on the integrand function and can be used as a general tool in treating uniform Airy-type expansions. The method is mainly of theoretical interest and delivers only order estimates for the remainders. In §8 we describe a method for obtaining strict error bounds for remainders of Airy-type expansions.

Our methods are not restricted to Airy-type expansions. In §7 we consider some other types of uniform expansions. In particular, a uniform expansion in terms of Bessel functions is considered. In this case the extension of the domain can be obtained, as can the double asymptotic property.

2. Related and earlier results. Airy-type expansions occur in the asymptotic theory of differential equations, for instance in turning-point problems; see [10, chap. 11]. In this case the estimation of remainders in terms of realistic and strict error bounds is well developed. Moreover, Olver extended the domains of the large parameter and the analogue of the saddle-point parameter to large areas in the complex plane.

The situation for integrals is quite different. Although the uniform Airy-type expansions have been extensively studied, a general theory for obtaining computable strict and realistic error bounds is still missing. This problem is more difficult than that for the case of differential equations. In transforming a given integral to a standard form by means of a mapping $x \mapsto w$ as in (1.2), a mapping $\alpha \mapsto b$ is implicitly introduced. Because of these two mappings, the function $h_0(w)$ in (1.3) may be difficult to handle. In corresponding problems in differential equations only the mapping $\alpha \mapsto b$ (or a related one) has to be considered. Another point is that in the theory of differential equations several techniques for bounding the remainders exist, but these techniques cannot be translated to the treatment of remainders of expansions obtained through integrals. An example is Olver's method that is based on bounding the remainders by using Volterra integral equations.

In [3] the analytical properties of the mapping (1.2) are considered locally around the relevant saddle points; in Friedman [8] another proof is given. Levinson [9] gives a fundamental mapping theorem that generalizes the mapping (1.2) considerably; see also [19, §7.6]. In Qu and Wong [11] an iterative method is used for proving the local analytic property of mappings that are more complicated than those defined by (1.2) (there is a pole in the neighborhood of the coalescing saddle points). The transformation (1.2) is discussed in terms of conformal mappings on unbounded domains for special cases; for instance, in Copson [4] for an integral defining the Bessel function $J_\nu(z)$, in [10] for the Anger function (a function related to the Bessel function), and in [7] for integrals defining the Laguerre polynomials.

Recent examples of the construction of strict bounds in uniform asymptotic expansions of integrals are presented in Shivakumar and Wong [12] and in Frenzen [5] for Legendre-function expansions and in Frenzen and Wong [6] for Jacobi polynomials. The expansions are not of the compound type that follows from the Bleistein method, and a restricted number of terms in the expansions are considered. Another approach is given in Ursell [17] for Legendre functions, where uniform bounds are obtained by applying the maximum-modulus theorem. Ursell's method does not give sharp computable estimates of the remainders, and extension of the bounded domain of z to an unbounded domain is indicated without proof. In Ursell [18] the Airy-type expansion is discussed by using the maximum-modulus principle for complex values of the saddle-point parameter. The possibility of extending the validity to unbounded domains is mentioned again. Earlier, in Ursell [16], the Airy-type expansion is compared with the steepest-descent expansion, giving a continuation to unbounded domains. Qualitative results are obtained for the coefficient functions and the remainders; the Bleistein sequence is not used.

In the Anger function example in [10] the region of the saddle-point parameter is extended to an unbounded real domain by giving order estimates of the remainder. Olver's technique is based on estimating remainders of Taylor series. The expansion is not of the Bleistein type but is obtained by expanding the integrand function at a saddle point inside the interval of integration. The analysis shows that the distance between singular points of the integrand function and that saddle point plays a crucial role, although the singularities are not mentioned explicitly.

In the treatment of Laguerre polynomials in [7] order estimates for the remainders are also given, and there is a claim of uniform validity with respect to the saddle-point parameter in an unbounded real domain. The claim does not follow from investigating the singularities of the integrand function. In the present paper we take into account the singularities, and we show that the claim is indeed correct.

Soni and Soni [14] give new representations of the coefficients and remainders of Airy-type expansions; these representations are based on an expansion of the integrand function in terms of a class of polynomials. The paper is a continuation of earlier papers by Soni and Sleeman [13] and Soni and Temme [15]. The coefficients and remainder are written as contour integrals of the integrand function and rational functions related with the polynomials. New order estimates of the remainder have been derived for a finite domain of the saddle-point parameter.

3. Uniform Airy-type expansion. Let

$$F(z,b) = \frac{1}{2\pi i}\int_{\mathcal{L}} e^{z(\frac{1}{3}w^3 - b^2 w)} h_0(w)\,dw, \tag{3.1}$$

where $h_0(w)$ is an analytic function on a neighborhood of $\mathcal{L}$, with $\mathcal{L}$ a suitable contour that begins at $\infty\exp(-\frac{1}{3}\pi i)$ and ends at $\infty\exp(\frac{1}{3}\pi i)$. When $b \in [0,\infty)$ we take $\mathcal{L}$ the steepest-descent contour through b, which is given by $\mathcal{L} = \{w = x + iy \in \mathbb{C} \mid y^2 = 3x^2 - 3b^2\}$ (see Fig. 3.1), such that $\mathrm{Im}(\frac{1}{3}w^3 - b^2 w) = 0$ and $\frac{1}{3}w^3 - b^2 w$ attains its maximum on $\mathcal{L}$ at b.

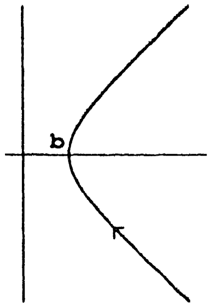

FIG. 3.1. *Steepest-descent curve $\mathcal{L}$ when $b \in [0,\infty)$.*

When $b \in [0, i\infty)$ we take $\mathcal{L} = \{w = x + iy \in \mathbb{C} \mid 3yx^2 = (y \pm ib)^2(y \mp 2ib)\}$, the steepest-descent contour through $\pm b$ (see Fig. 3.2).

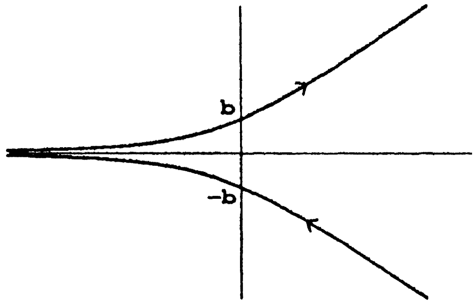

FIG. 3.2. *Steepest-descent curve $\mathcal{L}$ when $b \in [0, i\infty)$.*

It is not necessary to restrict our analysis to these contours of integration, but using these steepest-descent contours makes the following calculations less complicated.

We use Bleistein's method for obtaining an asymptotic expansion, defining $g_n(w)$, $h_{n+1}(w)$, $n = 0,1,2,\ldots$, by writing

$$\begin{aligned} h_n(w) &= \alpha_n + \beta_n w + (w^2 - b^2)g_n(w), \\ h_{n+1}(w) &= \frac{d}{dw}g_n(w), \end{aligned} \tag{3.2}$$

with α_n, β_n following from substitution of $w = \pm b$. If we use (3.2) in (3.1) and integrate n times by parts, we obtain

$$F(z,b) = \mathrm{Ai}(z^{2/3}b^2)\sum_{k=0}^{n-1}(-1)^k\alpha_k z^{-k-1/3} - \mathrm{Ai}'(z^{2/3}b^2)\sum_{k=0}^{n-1}(-1)^k\beta_k z^{-k-2/3} + \varepsilon_n, \tag{3.3}$$

where

$$\varepsilon_n = (-1)^n z^{-n}\frac{1}{2\pi i}\int_{\mathcal{L}} e^{z(\frac{1}{3}w^3 - b^2 w)}h_n(w)\,dw \tag{3.4}$$

and where $\mathrm{Ai}(z)$ is the Airy function and $\mathrm{Ai}'(z)$ is its derivative. The functions $h_n(w)$ share, by inheritance, the analytic properties of h_0 on the same neighborhood of $\mathcal{L}$.

Estimates of $|\varepsilon_n|$, for large values of z and for $|b|$ bounded, given in the literature are usually of the form

$$|\varepsilon_n| \le \frac{M_n}{z^{n+1/3}}\widetilde{\alpha}_n(b)|\widetilde{\mathrm{Ai}}(z^{2/3}b^2)| + \frac{N_n}{z^{n+2/3}}\widetilde{\beta}_n(b)|\widetilde{\mathrm{Ai}}'(z^{2/3}b^2)|, \tag{3.5}$$

where M_n and N_n depend on n and where $\widetilde{\alpha}_n$, $\widetilde{\beta}_n$ are related to the coefficients in (3.2). Furthermore,

$$\begin{aligned} \widetilde{\mathrm{Ai}}(u) &= \begin{cases} \mathrm{Ai}(u) & \text{if } u \ge 0, \\ \left[\mathrm{Ai}^2(u) + \mathrm{Bi}^2(u)\right]^{1/2} & \text{if } u < 0, \end{cases} \\ \widetilde{\mathrm{Ai}}'(u) &= \begin{cases} \mathrm{Ai}'(u) & \text{if } u \ge 0, \\ \left[\mathrm{Ai}'^2(u) + \mathrm{Bi}'^2(u)\right]^{1/2} & \text{if } u < 0. \end{cases} \end{aligned} \tag{3.6}$$

A proof of an estimate like (3.5) is given in [7], with

$$\widetilde{\alpha}_n(b) = \begin{cases} 1 & \text{if } 0 < b < \xi, \\ |\alpha_n| & \text{if } b > \xi, \end{cases} \qquad \widetilde{\beta}_n(b) = \begin{cases} 1 & \text{if } 0 < b < \xi, \\ |\beta_n| & \text{if } b > \xi, \end{cases}$$

where ξ is a fixed positive number.

Notice that the influence of large $|b|$ in (3.5) is not clear. We assume that the function h_0 of (3.1) depends on the saddle-point parameter b. Usually this is a consequence of the transformation to the standard form (3.1) by the mapping defined in (1.2). Also, when h_0 does not depend on b, all functions h_n obtained by recursion from (3.2) do depend on b.

For bounded $|b|$ an estimate like (3.5) holds for rather mild conditions on h_0. However, for obtaining uniformly valid estimates when b runs through an unbounded interval, we need more information on h_0. In the following sections we obtain estimates of $|h_n(w)|$ by formulating conditions on h_0 on discs with centers $\pm b$. These discs have radius $\rho(b)$, which indeed may be a function of b.

For obtaining estimates of ε_n of (3.4) holding in unbounded b-intervals, we now introduce a new class of rational functions.

4. Intermezzo: A new class of rational functions. We introduce a class of rational functions that satisfy the following theorem.

THEOREM 4.1. *Let*

$$R_0(u,w,b) = \frac{1}{u-w}, \tag{4.1.a}$$

$$R_{n+1}(u,w,b) = \frac{-1}{u^2-b^2}\frac{d}{du}R_n(u,w,b), \qquad n = 0,1,2,\ldots, \tag{4.1.b}$$

where $u, w, b \in \mathbb{C}$, $u \ne w$, $u^2 \ne b^2$. Let $h_n(w)$ be defined by the recursive scheme (3.2), *with $h_0(w)$ a given analytic function in a domain G. Then we have*

$$h_n(w) = \frac{1}{2\pi i}\int_C R_n(u,w,b)h_0(u)\,du, \tag{4.2}$$

where C is a simple closed contour in G that encircles the points w and $\pm b$.

Proof.

$$\begin{aligned} h_n(w) &= \frac{1}{2\pi i}\int_C R_0(u,w,b)h_n(u)du = \frac{1}{2\pi i}\int_C R_0(u,w,b)\frac{d}{du}g_{n-1}(u)du \\ &= \frac{1}{2\pi i}\int_C R_1(u,w,b)h_{n-1}(u)du - \frac{1}{2\pi i}\int_C R_1(u,w,b)(\alpha_{n-1}+\beta_{n-1}u)du \\ &=_* \frac{1}{2\pi i}\int_C R_1(u,w,b)h_{n-1}(u)du \\ &\ \vdots \\ &= \frac{1}{2\pi i}\int_C R_n(u,w,b)h_0(u)du. \end{aligned}$$

In * we use the fact that the rational function $R_1(u,w,b)(\alpha_{n-1}+\beta_{n-1}u)$ is $\mathcal{O}(u^{-2})$ as $|u| \to \infty$ and that all the poles of this function are inside C. Thus the integral of this function along C vanishes (use the transformation $u \mapsto u^{-1}$, which is well defined at $u = \infty$ and yields an integral with no singularities inside the contour of integration). □

COROLLARY 4.2. *Let $A_n(u,b)$, $B_n(u,b)$ be defined by the recursion in* (4.1.b), *with initial values*

$$A_0(u,b) = \frac{u}{u^2-b^2}, \qquad B_0(u,b) = \frac{1}{u^2-b^2}. \tag{4.3}$$

Then for $n = 0,1,2,\ldots$, the coefficients α_n, β_n of (3.2) *can be written as*

$$\alpha_n = \frac{1}{2\pi i}\int_C A_n(u,b)h_0(u)du, \qquad \beta_n = \frac{1}{2\pi i}\int_C B_n(u,b)h_0(u)du, \tag{4.4}$$

where C is a simple closed contour in G that encircles the points $\pm b$.

We observe that the rational functions defined by (4.1) are independent of the function h_0 and that representation (4.2) can be considered as the analogue of the Cauchy integral defining the remainder of a Taylor series. An estimate of h_n, the integrand function of (3.4), will be obtained as in Cauchy's inequality for bounding the coefficients of a Taylor series.

By induction with respect to n, it follows that R_n has an expansion of the form

$$R_n(u,w,b) = \sum_{i=0}^{n-1}\sum_{j=0}^{k_{n,i}} \frac{C_{ij}u^{i-j}}{(u-w)^{n+1-i-j}(u^2-b^2)^{n+i}}, \qquad n = 1,2,\ldots, \tag{4.5}$$

with $k_{n,i} = \min(i, n-1-i)$ and where C_{ij} do not depend on u, w, and b.

We conclude this section by giving estimates for R_n and for integrals of this function; these can be proved easily with (4.5).

(i) Let $w \in \mathbb{C}$ such that $|w - b| = \mathcal{O}(b)$ as $b \to \infty$, and let Γ be a simple closed contour that encircles b and w. Then for $n = 1,2,\ldots$,

$$\frac{1}{2\pi i}\int_\Gamma R_n(u,w,b)du = \mathcal{O}(b^{-3n}) \tag{4.6}$$

as $b \to \infty$.

(ii) Let $b \in \mathbb{C}$ and $\Omega(b) = \{(u,w) \in \mathbb{C}^2 \mid |u-b| = \rho(b),\ |w-b| \le \frac{1}{2}\rho(b)\}$, such that $\rho(b) = \mathcal{O}(|b|^\theta)$ as $b \to \infty$, where $-\frac{1}{2} < \theta \le 1$. Then we can assign numbers A_n independent of b, such that

$$\sup_{(u,w)\in\Omega(b)} |R_n(u,w,b)| \le A_n |b|^{-(1+2\theta)n-\theta} \quad \text{as} \quad b \to \infty. \tag{4.7}$$

5. Extension of the domain of validity. In this section we prove that, under certain circumstances, expansion (3.3) holds uniformly with respect to the saddle-point parameter b in unbounded domains.

For defining the radius $\rho(b)$ of the discs mentioned at the end of §3, we first define

$$\rho_0(b) = \min\{|w \pm b| \mid w \text{ is a singularity of } h_0(w)\} \tag{5.1}$$

and we assume that, for large $|b|$, we have $\rho_0(b) \ge \delta_0 |b|^\theta$, where the constants δ_0 and θ satisfy $\delta_0 > 0$, $\theta > -\frac{1}{2}$. This is the essential assumption on $h_0(w)$ in the neighborhood of the saddle points.

Now we take $\rho(b) \le \rho_0(b)$ such that $\rho(b) \sim \delta|b|^\theta$ as $b \to \infty$, where the constant $\delta > 0$. We take $\theta \le 1$ as large as possible, and we drop the restriction $\theta \le 1$ after Theorem 5.2. Notice that we concentrate on estimates with $|b| \to \infty$ and that we do not give details for b in compacta.

Next we introduce upper bounds for the $h_n(w)$, $n = 0, 1, 2, \ldots$. Thus let

$$\widetilde{h}_n = \sup_{|w\pm b| \le (1/2)\rho(b)} |h_n(w)|. \tag{5.2}$$

Notice that $h_0(w)$ is analytic on $|w \pm b| < \rho(b)$; thus $\widetilde{h}_0$ is finite.

For obtaining estimates of $\widetilde{h}_n$ in terms of $\widetilde{h}_0$ let Γ be a circle around $\pm b$ with radius $\rho(b)$ and let $|w \mp b| \le \frac{1}{2}\rho(b)$. We require $\theta \le 1$ to ensure that both saddle points are not inside the circle Γ. This is possible by choosing δ appropriately. Then, if we use (4.6), we have

$$\begin{aligned}
h_n(w) &= \frac{1}{2\pi i}\int_\Gamma R_0(u,w,b)h_n(u)\,du \\
&= \frac{1}{2\pi i}\int_\Gamma R_1(u,w,b)h_{n-1}(u)\,du - \frac{1}{2\pi i}\int_\Gamma R_1(u,w,b)(\alpha_{n-1} + \beta_{n-1}u)\,du \\
&=_{(4.6)} \frac{1}{2\pi i}\int_\Gamma R_1(u,w,b)h_{n-1}(u)\,du + \widetilde{h}_{n-1}\mathcal{O}(b^{-3}) \\
&\ \vdots \\
&=_{(4.6)} \frac{1}{2\pi i}\int_\Gamma R_n(u,w,b)h_0(u)\,du + \widetilde{h}_{n-1}\mathcal{O}(b^{-3}) + \cdots + \widetilde{h}_0\mathcal{O}(b^{-3n})
\end{aligned}$$

as $b \to \infty$. So by induction and (4.7) we have proved the following theorem.

THEOREM 5.1. *Let $\widetilde{h}_n$, $n = 0, 1, 2, \ldots$, be the upper bound of $h_n(w)$, defined in (5.2). Then we have the estimate*

$$\widetilde{h}_n \le C_n |b|^{-(1+2\theta)n}\widetilde{h}_0 \quad \text{as} \quad b \to \infty, \tag{5.3}$$

where C_n does not depend on b.

Now we shall prove that ε_n can be bounded as follows:

$$|\varepsilon_n| \le C_n(|b|+1)^{-(1+2\theta)n}\widetilde{h}_0 z^{-n-1/3}\widetilde{\mathrm{Ai}}(z^{2/3}b^2), \tag{5.4}$$

with a slightly different C_n that does not depend on b and z.

In order to use the preceding estimates, we split up the contour $\mathcal{L}$ into $\mathcal{L}'$ and $\mathcal{L}''$. In the case that $b \in [0,\infty)$ we take $\mathcal{L}' = \{w \in \mathcal{L} \mid |w-b| \le \frac{1}{2}\rho(b)\}$, and in the case that $b \in [0, i\infty)$ we take $\mathcal{L}' = \{w \in \mathcal{L} \mid |w \pm b| \le \frac{1}{2}\rho(b)\}$. We define $\mathcal{L}'' = \mathcal{L} - \mathcal{L}'$ and introduce the corresponding integrals:

$$\begin{aligned} \varepsilon_{n|_{\mathcal{L}'}} &= (-1)^n z^{-n}\frac{1}{2\pi i}\int_{\mathcal{L}'} e^{z(\frac{1}{3}w^3-b^2w)}h_n(w)\,dw, \\ \varepsilon_{n|_{\mathcal{L}''}} &= (-1)^n z^{-n}\frac{1}{2\pi i}\int_{\mathcal{L}''} e^{z(\frac{1}{3}w^3-b^2w)}h_n(w)\,dw. \end{aligned} \tag{5.5}$$

In the Appendix we formulate conditions on $h_0(w)$ such that when $\theta > -\frac{1}{2}$ the estimate of $|\varepsilon_{n|_{\mathcal{L}''}}|$ is exponentially small compared with the estimate of $|\varepsilon_{n|_{\mathcal{L}'}}|$ as $z \to \infty$ uniformly with respect to b.

The proof of (5.4) for large b is divided into separate cases: (i) $b \in [0,\infty)$ and (ii) $b \in [0, i\infty)$. We first consider case (i). With (5.3) we have

$$\begin{aligned} |\varepsilon_{n|_{\mathcal{L}'}}| &\le C_n z^{-n}|b|^{-(1+2\theta)n}\widetilde{h}_0\frac{1}{2\pi i}\int_{\mathcal{L}'} e^{z(\frac{1}{3}w^3-b^2w)}dw \\ &\le C_n z^{-n-1/3}|b|^{-(1+2\theta)n}\widetilde{h}_0\mathrm{Ai}(z^{2/3}b^2). \end{aligned}$$

In case (ii) we write $w = x+iy$ and we define $\mathcal{L}'_+ = \{y > 0 \mid$ there exists $x \in \mathbb{R} : x+iy \in \mathcal{L}'\}$. Simple transformations give

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\mathcal{L}'} e^{z(\frac{1}{3}w^3-b^2w)}h_n(w)\,dw = \\ &\quad \frac{1}{2\pi i}\int_{\mathcal{L}'_+} e^{-z(y+ib)^2 f(y)}g(y)(e^{-\frac{2}{3}zb^3}h_n(w) - e^{+\frac{2}{3}zb^3}h_n(\overline{w}))dy \\ &\quad + \frac{1}{2\pi}\int_{\mathcal{L}'_+} e^{-z(y+ib)^2 f(y)}(e^{-\frac{2}{3}zb^3}h_n(w) + e^{+\frac{2}{3}zb^3}h_n(\overline{w}))dy, \end{aligned}$$

where

$$f(y) = \frac{2(2y-ib)^2}{9y}\sqrt{\frac{y-2ib}{3y}}, \qquad g(y) = \sqrt{\frac{y-2ib}{3y}} + \frac{ib(y+ib)}{3y^2}\sqrt{\frac{3y}{y-2ib}}.$$

Note that the functions have real arguments and that $g(y) \ge 0$. Thus with (5.3) we have

$$\begin{aligned} &\left|\frac{1}{2\pi i}\int_{\mathcal{L}'} e^{z(\frac{1}{3}w^3-b^2w)}h_n(w)\,dw\right| \\ &\le \frac{1}{2\pi}\int_{\mathcal{L}'_+} e^{-z(y+ib)^2 f(y)}(1+g(y))(|h_n(w)|+|h_n(\overline{w})|)dy \\ &\le_{(5.3)} C_n|b|^{-(1+2\theta)n}\widetilde{h}_0\frac{1}{\pi}\int_0^\infty e^{-z(y+ib)^2 f(y)}(1+g(y))dy \\ &\le C'_n|b|^{-(1+2\theta)n}\widetilde{h}_0\frac{1}{\sqrt{zb/i}} \\ &\sim_* \pi^{1/2}C'_n|b|^{-(1+2\theta)n}\widetilde{h}_0 z^{-1/3}\widetilde{\mathrm{Ai}}(z^{2/3}b^2), \end{aligned}$$

as $z \to \infty$. In * we have used the relation $\widetilde{\mathrm{Ai}}(x) \sim \pi^{-1/2}(-x)^{-1/4}$ as $x \to -\infty$; see [10, p. 395].

In the Appendix we prove that

$$|\varepsilon_n|_{\mathcal{L}''}| \le C_n e^{-\lambda(z-\mu)|b|^{2\theta+1}} \widetilde{h}_0 z^{-n-4/3} \widetilde{\mathrm{Ai}}(z^{2/3}b^2), \tag{5.6}$$

where the positive C_n, λ, and μ do not depend on b and z and where $|b| \ge c > 0$. These estimates show that (5.4) is valid. Thus we have proved the following theorem.

THEOREM 5.2. *Let $F(z,b)$ be of the form (3.1), where $h_0(w)$ satisfies the conditions mentioned in the beginning of this section and in the Appendix. Then we have (3.3) as a uniform asymptotic expansion for $F(z,b)$, where (5.4) is an estimate for $|\varepsilon_n|$ as $z \to \infty$ uniformly with respect to $b \in [0,\infty) \cup [0, i\infty)$ and where $\widetilde{h}_0$ is given in (5.2).*

Now we drop the restriction $\theta \le 1$. In the case that $\theta > 1$, the analysis that leads to Theorem 5.2 is much easier; every time $1 + 2\theta$ occurs it can be replaced with the larger factor 3θ.

Remark 1. With the conditions of Theorem 5.2 it follows that expansion (3.3) has a double asymptotic property: the roles of b and z can be interchanged. The double asymptotic property is lost in the example considered in §6.

Remark 2. An estimate like (5.4) has been derived in [10, p. 360] for a particular example. There the estimate for the remainder of an expansion of the Anger function $\mathrm{A}_{-\nu}(\nu \operatorname{sech} \alpha)$ reads

$$\varepsilon_n(\alpha, \nu) = (1+\xi)^{-\theta(n+1)} \nu^{-\frac{1}{3}(n+1)} \mathrm{Qi_n}(\nu^{\frac{2}{3}}\xi) \mathcal{O}(1),$$

where $\mathrm{Qi_n}(z)$ is a special function, $\frac{2}{3}\xi^{3/2} = \alpha - \tanh \alpha$, and $\theta = -\frac{1}{4}$. This estimate holds as $\nu \to \infty$ uniformly with respect to $\alpha \in [0,\infty)$ or $\xi \in [0,\infty)$. Indeed, the value $\theta = -\frac{1}{4}$ is related to the distance between the relevant saddle point and the nearest singularities of the integrand function, which is of order $\xi^{-1/4}$ as $\xi \to \infty$.

6. Laguerre polynomials: A boundary case. In this section we show that, in certain circumstances, the condition $\theta > -\frac{1}{2}$ of Theorem 5.2 can be replaced with $\theta = -\frac{1}{2}$. We demonstrate this feature by considering a recent expansion for the Laguerre polynomials.

First we summarize the main steps for obtaining an Airy-type expansion of the Laguerre polynomials. More details are given in [7] and [19]. Laguerre polynomials have the following integral representation:

$$(-1)^N 2^\alpha e^{-zt/2} L_N^{(\alpha)}(zt) = \frac{1}{2\pi i} \int_{+\infty}^{(1+)} e^{zf(x,t)} (1-x^2)^{\frac{\alpha-1}{2}}\, dx, \tag{6.1}$$

where the contour of integration begins and ends at $+\infty$ and encircles 1 in the positive direction and where

$$f(x,t) = \frac{1}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{2}xt \tag{6.2}$$

and $z = 4N + 2\alpha + 2$, $\alpha > -1$, and $t \ge 1$. Again, we use the transformation

$$f(x,t) = \tfrac{1}{3}w^3 - b^2 w. \tag{6.3}$$

The x-saddle points $\pm\sqrt{1-1/t}$ should correspond with $\pm b$. It follows that

$$b^3 = \tfrac{3}{4}\left(\sqrt{t^2-t} - \operatorname{arccosh}\sqrt{t}\right). \tag{6.4}$$

With transformation (6.3) we have for (6.1)

$$(-1)^N 2^\alpha e^{-zt/2} L_N^{(\alpha)}(zt) = \frac{1}{2\pi i}\int_{\mathcal{L}} e^{z(\frac{1}{3}w^3 - b^2 w)} h_0(w)\,dw, \tag{6.5}$$

where

$$h_0(w) = (1-x^2)^{\frac{\alpha-1}{2}}\frac{dx}{dw} = 2\frac{(1-x^2)^{\frac{\alpha+1}{2}}(w^2-b^2)}{1-t(1-x^2)} \tag{6.6}$$

and $\mathcal{L}$ is given in Fig. 3.1. Again, using (3.2) in (6.5), we obtain

$$\begin{aligned}(-1)^N 2^\alpha e^{-zt/2} L_N^{(\alpha)}(zt) = {}& \mathrm{Ai}(z^{2/3}b^2)\sum_{k=0}^{n-1}(-1)^k \alpha_k z^{-k-1/3} \\ & - \mathrm{Ai}'(z^{2/3}b^2)\sum_{k=0}^{n-1}(-1)^k \beta_k z^{-k-2/3} + \varepsilon_n,\end{aligned} \tag{6.7}$$

where ε_n is as in (3.4).

To apply the analysis of §5, we locate the relevant singular points of $h_0(w)$. Let $x_0 = \sqrt{1-1/t}$ be the positive x-saddle point when $t > 1$. The point x_0 is mapped to $w(x_0) = b$ by the mapping given in (6.3), when the logarithmic function takes its principal value. However, the points x_0 at other sheets of the Riemann surface of the log function are singular points of the mapping (6.3). Then the phase of $1 - x_0$ is, for instance, 2π. When $b = 0$ the singularities $w = S_\pm$ nearest to b satisfy $\frac{1}{3}S_\pm^3 = \pm\frac{1}{2}\pi i$, whereas

$$S_\pm \sim b \pm \sqrt{\frac{\pi i}{2b}} \quad \text{as} \quad b \to \infty. \tag{6.8}$$

Thus $\rho_0(b)$ of (5.1) is of order $b^{-1/2}$ as $b \to \infty$.

As before, we want to split up $\mathcal{L}$ into $\mathcal{L}'$ and $\mathcal{L}''$, and define $\varepsilon_{n|_{\mathcal{L}'}}$, $\varepsilon_{n|_{\mathcal{L}''}}$ similar to (5.5). So define $\mathcal{L}' = \{w \in \mathcal{L} \mid |w-b| \le \delta b^\theta\}$, where the constants δ and θ satisfy $\delta > 0$ and $-\frac{1}{2} < \theta \le 1$, in order that the estimate of $|\varepsilon_{n|_{\mathcal{L}''}}|$ is exponentially small compared with the estimate of $|\varepsilon_{n|_{\mathcal{L}'}}|$ as $z \to \infty$ uniformly with respect to b. We choose θ close to $-\frac{1}{2}$ fixed.

Let Γ_θ be a closed contour which encircles $\mathcal{L}'$ such that

$$\text{length}\,\Gamma_\theta = \mathcal{O}(b^\theta), \quad \text{distance}(\Gamma_\theta, \mathcal{L}') \sim cb^{-1/2} \quad \text{as} \quad b \to \infty$$

and such that $h_0(w)$ is analytic on $\overline{I(\Gamma_\theta)}$, where $\overline{I(\Gamma_\theta)}$ is the closure of the interior of Γ_θ. Then straightforward calculations give that

$$\sup_{w \in \overline{I(\Gamma_\theta)}} |h_0(w)| \le C_0 b^{(\theta+\frac{1}{2})\alpha}|h_0(b)| \quad \text{as} \quad b \to \infty, \tag{6.9}$$

where C_0 does not depend on b. Further calculations, similar to those in §5 yield, for $n = 1, 2, \ldots,$

$$\sup_{w\in\overline{I(\Gamma_\theta)}} |h_n(w)| \le C_n b^{(\theta+\frac{1}{2})(\alpha+1)}|h_0(b)| \quad \text{as} \quad b \to \infty, \tag{6.10}$$

where, here and below, C_n denotes a generic quantity that does not depend on b and z. Notice that, in contrast to (5.3), the power of b is positive and does not depend on n. These estimates yield

$$|\varepsilon_{n|_{\mathcal{L}'}}| \le C_n z^{-n-1/3} b^{(\theta-1)\alpha+(\theta-\frac{1}{2})}\mathrm{Ai}(z^{2/3}b^2). \tag{6.11}$$

In (6.11) we used

$$h_0(b) = t^{\frac{(1-\alpha)}{2}} \frac{\sqrt{2b}}{(t-1)^{1/4}t^{3/4}}.$$

In the Appendix we prove that

$$|\varepsilon_{n|_{\mathcal{L}''}}| \le C_n' z^{-n-4/3} e^{-\lambda(z-2)b^{2\theta+1}} |h_0(b)|\mathrm{Ai}(z^{2/3}b^2), \tag{6.12}$$

where the positive C_n' and λ do not depend on b and z. It follows that we can assign numbers C_n, independent of z and b, such that

$$|\varepsilon_n| \le C_n z^{-n-1/3}(b+1)^{(\theta-1)\alpha+(\theta-\frac{1}{2})}\mathrm{Ai}(z^{2/3}b^2), \tag{6.13}$$

as $z \to \infty$ uniformly with respect to $b \in [0, \infty)$. A similar approach can be used for $b \in [0, i\tau]$, where $0 < \tau < (\frac{3}{8}\pi)^{1/3}$, τ fixed.

We can compare this estimate with the estimate given in [7] and [19], which is of the form (3.5). First, we notice that (6.13) is not in the form of the first neglected terms of expansion (6.7). But with (6.10) it easily follows that the first neglected terms can be estimated by the right-hand side of (6.13). Regardless, (6.13) clearly shows why expansion (6.7) holds uniformly with respect to b in an unbounded domain. Secondly, in (6.13) the influence of b is more transparent than in the right-hand side of (3.5).

7. Other uniform expansions generated by the Bleistein method. In this section we show that the methods used for the Airy-type expansions are quite general and can be applied to other uniform expansions of integrals of the form

$$\int_{\mathcal{C}} e^{zf(x,b)} h_0(x)\,dx \tag{7.1}$$

with coinciding saddle points and singularities. In this section we work out an example of uniform expansions in terms of Bessel functions. In [7] such an expansion of the Laguerre polynomials is given. Let

$$F(z, A) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} w^{-\alpha-1} h_0(w) e^{\frac{1}{2}z(w-A^2/w)}\,dw, \tag{7.2}$$

where the contour of integration begins and ends at $-\infty$ and encircles the origin in the positive direction. We assume that $h_0(w)$ is analytic on a neighborhood of the contour of integration, and we let $z > 0$, $iA > 0$, and $\alpha > -1$. Notice that $\pm iA$ are

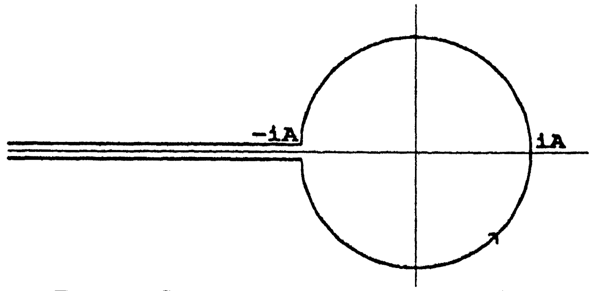

FIG. 7.1. *Steepest-descent curve for integral* (7.2).

the saddle points of the integral. We choose the contour of integration through these saddle points, and the steepest-descent path looks like Fig. 7.1.

The recursion in connection with integral (7.2) is

(7.3)
$$h_n(w) = \alpha_n + \frac{\beta_n}{w} + \left(1 + \frac{A^2}{w^2}\right) g_n(w), \qquad h_{n+1}(w) = w^{\alpha+1}\frac{d}{dw}\left[w^{-\alpha-1} g_n(w)\right],$$

and if we integrate n times by parts, we obtain the expansion

$$F(z,A) = \frac{J_\alpha(zA)}{A^\alpha}\sum_{k=0}^{n-1}(-1)^k\alpha_k\left(\frac{2}{z}\right)^k + \frac{J_{\alpha+1}(zA)}{A^{\alpha+1}}\sum_{k=0}^{n-1}(-1)^k\beta_k\left(\frac{2}{z}\right)^k + \varepsilon_n, \tag{7.4}$$

where

$$\varepsilon_n = (-1)^n\left(\frac{2}{z}\right)^n \frac{1}{2\pi i}\int_{-\infty}^{(0+)} w^{-\alpha-1}h_n(w)e^{\frac{1}{2}z(w-A^2/w)}dw \tag{7.5}$$

and where $J_\alpha(z)$ and $J_{\alpha+1}(z)$ are Bessel functions of the first kind. Since zA is purely imaginary, modified Bessel functions occur in the expansion.

The class of rational functions generated by (7.3) is recursively defined by

$$\begin{aligned} Q_0(u,w,A) &= \frac{1}{u-w}, \\ Q_{n+1}(u,w,A) &= \frac{-1}{1+A^2/u^2}\left(\frac{\alpha+1}{u} + \frac{d}{du}\right)Q_n, \qquad n = 0,1,2,\ldots. \end{aligned} \tag{7.6}$$

By induction with respect to n it follows that Q_n has an expansion of the form

$$Q_n(u,w,A) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-i}\frac{C_{ij}(A^2/u^2)^i}{(u-w)^{n+1-i-j}u^{i+j}(1+A^2/u^2)^{n+i}}, \qquad n = 1,2,\ldots, \tag{7.7}$$

where the C_{ij} do not depend on u, w, and A.

Again, we concentrate on the influence of A on the expansion (7.4), especially when $|A|$ is large.

If Γ is a simple closed contour that encircles iA and w and with $-iA$ in its exterior, then we can prove, just as for (4.6), that

$$\frac{1}{2\pi i}\int_\Gamma Q_n(u,w,A)du = \mathcal{O}(|A|^{-n}) \quad \text{as} \quad |A| \to \infty. \tag{7.8}$$

As before, we want to split up $\mathcal{L}$ into $\mathcal{L}'$ and $\mathcal{L}''$. We assume that, for large $|A|$, the distance from the singularities of $h_0(w)$ to the saddle points $\pm iA$ is at least $\delta|A|^\theta$, where the constants δ, θ satisfy $\delta > 0$, $\frac{1}{2} < \theta \le 1$. Consequently, we take $\mathcal{L}' = \{w \in \mathcal{L} \mid |w - iA| \le \frac{1}{2}\delta|A|^\theta\}$ and $\mathcal{L}'' = \mathcal{L} - \mathcal{L}'$ such that the estimate of $|\varepsilon_{n|_{\mathcal{L}''}}|$ is exponentially small compared with the estimate of $|\varepsilon_{n|_{\mathcal{L}'}}|$ as $z \to \infty$ uniformly with respect to $iA \in [c, \infty)$, where $c > 0$ fixed. In fact, we need a growth condition on $h_0(w)$ on a prescribed neighborhood of $\mathcal{L}''$, which is similar to the condition mentioned in the Appendix.

If we set $\Omega(A) = \{(u, w) \in \mathbb{C}^2 \mid |u - iA| = \frac{3}{4}\delta|A|^\theta,\ |w - iA| \le \frac{1}{2}\delta|A|^\theta\}$, we can prove

$$\sup_{(u,w)\in\Omega(A)} |Q_n(u, w, A)| \le C_n|A|^{(1-2\theta)n-\theta}, \tag{7.9}$$

where C_n does not depend on A. Finally, we define

$$\widetilde{h}_0 = \sup_{|w\pm iA|\le(1/2)\delta|A|^\theta} |h_0(w)|. \tag{7.10}$$

With (7.8), (7.9), and straightforward calculations similar to those leading to (5.3), we obtain, for $n = 1, 2, \ldots$,

$$\sup_{|w-iA|\le(1/2)\delta|A|^\theta} |h_n(w)| \le C_n|A|^{(1-2\theta)n}\widetilde{h}_0 \quad \text{as} \quad |A| \to \infty, \tag{7.11}$$

where C_n does not depend on A. With the aid of these estimates we obtain as the main result of this section

$$|\varepsilon_n| \le C_n(|A| + 1)^{(1-2\theta)n-\alpha}\widetilde{h}_0 z^{-n}|J_\alpha(zA)| \tag{7.12}$$

as $z \to \infty$ uniformly with respect to $iA \in [0, \infty)$, where C_n does not depend on A and z.

In the case that $\theta > 1$ we can use the same analysis that leads to (7.12), but every time $1 - 2\theta$ occurs it has to be replaced with $-\theta$.

A similar approach is possible for real values of A.

8. Strict upper bounds of the remainder. In this section we assume that we have quantitative information on the functions $h_n(w)$ and that we can construct upper bounds for the remainders ε_n of (3.4). The simplest case is that we know that $|h_n(w)|$ is bounded on $\mathcal{L}$. If $b \ge 0$, an upper bound for ε_n can be easily expressed in terms of this bound and of the Airy function $\mathrm{Ai}(z^{2/3}b^2)$. When $b^2 < 0$ (the oscillatory case), the bound can be expressed in terms of the modulus function $\left[\mathrm{Ai}^2(z^{2/3}b^2) + \mathrm{Bi}^2(z^{2/3}b^2)\right]^{1/2}$ (see also (3.6)).

When the maximal value of $|h_n(w)|$ occurs at $w = w_0$, with w_0 far away from the saddle point $w = b$, the upper bound obtained in this way may be quite inaccurate. The fact is that the main contributions to the integral (3.4) come from a small neighborhood of b, especially when z is large. To obtain realistic upper bounds of $|\varepsilon_n|$ we describe a different approach in which we also allow unbounded functions $h_n(w)$. We concentrate on the case $b \ge 0$.

The contour $\mathcal{L}$ can be parameterized by writing $w = x + iy$, $3x^2 - y^2 = 3b^2$. By using this and integrating with respect to y, the integral (3.1) can be written in the form

$$\varepsilon_n = (-1)^n z^{-n} \frac{e^{-\frac{2}{3}zb^3}}{2\pi i} \int_{-\infty}^{\infty} e^{-z\phi(y)} H_n(y)\, dy, \tag{8.1}$$

where

$$\phi(y) = (\tfrac{8}{9}y^2 + \tfrac{2}{3}b^2)\sqrt{\tfrac{1}{3}y^2 + b^2} - \tfrac{2}{3}b^3, \qquad H_n(y) = h_n(x + iy)\left[\tfrac{dx}{dy} + i\right],$$

and

$$\frac{dx}{dy} = \frac{\frac{1}{3}y}{\sqrt{\frac{1}{3}y^2 + b^2}}.$$

When $h_0 = 1$ and $n = 0$ we obtain the real representation for the Airy function:

$$\text{(8.2)} \qquad \operatorname{Ai}(z^{\frac{2}{3}}b^2) = \frac{z^{\frac{1}{3}}e^{-\frac{2}{3}zb^3}}{2\pi}\int_{-\infty}^{\infty} e^{-z\phi(y)}\,dy.$$

To bound ε_n we assume that for fixed b the function $H_n(y)$ is majorized by

$$\text{(8.3)} \qquad |H_n(y)| \le M_n e^{\sigma_n \phi(y)}, \qquad -\infty < y < \infty,$$

where M_n and σ_n are nonnegative numbers that may depend on b. Observe that, in fact, only the even part of the function $H_n(y)$ needs to be bounded in this way; when $h_n(w)$ is a real function, the even part of $H_n(y)$ equals the imaginary part. The best strategy is to start with M_n and to define it slightly larger than $|H_n(0)| = |h_n(b)|$ (when this quantity vanishes a minor modification is needed), say, $M_n = 1.25|h_n(b)|$. Next we determine the smallest number σ_n that satisfies the upper bound in (8.3). When $|h_n(w)|$ is bounded on $\mathcal{L}$ and assumes its maximal value on $\mathcal{L}$ at $w = w_0 = x_0 + iy_0$, one may take $M_n = |H_n(y_0)|$ and $\sigma_n = 0$. However, as mentioned previously, when y_0 is not close to zero, the resulting bound may be unrealistic. When $\sigma_n > 0$, the argument of the exponential function in the right-hand side of (8.3) is unbounded; thus we accept unbounded functions $|h_n(w)|$. Observe that far away from the origin the estimate (8.3) may be very rough, but there the contribution to the integral (8.1) is negligible, especially when z is large.

Using (8.3) in (8.1) (when $z > \sigma_n$), we obtain with (8.2) the estimate

$$\text{(8.4)} \qquad |\varepsilon_n| \le M_n z^{-n}(z-\sigma_n)^{-\frac{1}{3}}\operatorname{Ai}\left((z-\sigma_n)^{\frac{2}{3}}b^2\right)e^{-\frac{2}{3}b^3\sigma_n}, \qquad z > \sigma_n, \quad b \ge 0.$$

The factor M_n contains the information on the parameter b; especially, it contains the information on whether or not the expansion holds uniformly on unbounded b-domains and has the double asymptotic property.

This bound is computable when the function $h_n(w), w \in \mathcal{L}$ is computable. Representation (4.2) may be helpful in computing $h_n(w)$. We expect that the bound in (8.4) is realistic for a wide class of functions $h_0(w)$.

When the function $h_n(w)$ grows too fast with b, the number σ_n may be an unbounded function of b. In that case the bound in (8.4) loses its uniform character. For example, when $h_0(w) = \exp(-w^2b^2)$ it is easily verified that the minimal value of σ_0 that satisfies (8.3) is $\sigma_0 = (2/243)b$.

When $b \in [0, i\infty)$ a similar approach is possible by majorizing the function $h_n(w)$ on the contour of Fig. 3.2. The analysis and the resulting bounds are slightly more complicated. Details will not be given.

9. An example. We consider the function

$$\text{(9.1)} \qquad F(z,b) = \frac{1}{2\pi i}\int_{\mathcal{L}} e^{z(\frac{1}{3}w^3 - b^2 w)}\frac{1}{w-b-1}\,dw,$$

where $b \in [0, \infty)$ and $\mathcal{L}$ is the steepest-descent contour shown in Fig. 3.1. $F(z, b)$ can be written as an integral of the Airy function, that is,

$$F(z,b) = e^{(b+1)\zeta}\left[-e^{\frac{1}{3}z(b+1)^3} + z^{-\frac{1}{3}}\int_\zeta^\infty e^{(b+1)t}\mathrm{Ai}(tz^{-\frac{1}{3}})\,dt\right], \qquad \zeta = zb^2.$$

In this example we have $h_0(w, b) = 1/(w - b - 1)$. Thus the quantities introduced in (5.1) and (5.2) are as follows: $\rho_0(b) = 1$, $\theta = 0$, and $\widetilde{h}_0 = 2$. It is easily verified that

$$h_1(w) = \frac{-1}{(2b+1)(w-b-1)^2}, \qquad h_1(b) = \frac{-1}{2b+1},$$
$$h_2(w) = 2\frac{b^2+4b+2-(b+1)w}{(2b+1)^3(w-b-1)^3}, \qquad h_2(b) = -2\frac{3b+2}{(2b+1)^3}.$$

Further calculations show that

$$\begin{aligned} F(z,b) &= \mathrm{Ai}(z^{2/3}b^2)\alpha_0 z^{-1/3} - \mathrm{Ai}'(z^{2/3}b^2)\beta_0 z^{-2/3} + \varepsilon_1 \\ &= \mathrm{Ai}(z^{2/3}b^2)\left(\alpha_0 - \frac{\alpha_1}{z}\right)z^{-1/3} - \mathrm{Ai}'(z^{2/3}b^2)\left(\beta_0 - \frac{\beta_1}{z}\right)z^{-2/3} + \varepsilon_2, \end{aligned} \tag{9.2}$$

with

$$\begin{aligned} \alpha_0 &= -\frac{b+1}{2b+1}, & \beta_0 &= -\frac{1}{2b+1}, \\ \alpha_1 &= -\frac{2b^2+2b+1}{(2b+1)^3}, & \beta_1 &= -2\frac{b^2+b}{(2b+1)^3}, \\ \alpha_2 &= -4\frac{3b^3+5b^2+3b+1}{(2b+1)^5}, & \beta_2 &= -2\frac{6b^2+10b+5}{(2b+1)^5}. \end{aligned} \tag{9.3}$$

We can determine the numbers M_n, σ_n occurring in (8.3), but already for this simple example optimal values have to be computed numerically. Analytical bounds of $\mathrm{Im}\, H_n(y)$ are easily obtained, however. For example, we have (recall that $x = \sqrt{(1/3)y^2 + b^2}$)

$$\mathrm{Im}\, H_0(y) = \mathrm{Im}\left[\left(i + \frac{dx}{dy}\right)\frac{1}{x+iy-b-1}\right] = \frac{b^2-(b+1)x}{x[(x-b-1)^2+3x^2-3b^2]}, \qquad x \ge b$$

(changing to x gives better formulas). When $b \ge \frac{1}{2}$ we have $|\mathrm{Im}\, H_0(y)| \le |h_0(b)|$; when $b \in [0, \frac{1}{2})$ the maximal value of $|\mathrm{Im}\, H_0(y)|$ is slightly larger than $|h_0(b)|$. Similar results hold for $n = 1, 2$, where the critical b-values are $b = \frac{1}{3}, b = (\sqrt{7} - 1)/6 = 0.27$, respectively. It follows that in this example the remainders can be estimated in terms of the first neglected terms of the asymptotic expansion (note that $h_n(b) = \alpha_n + b\beta_n$):

$$\begin{aligned} |\varepsilon_1| &\le |h_1(b)| z^{-4/3}\mathrm{Ai}(z^{2/3}b^2), & b &\ge \tfrac{1}{3}, \\ |\varepsilon_2| &\le |h_2(b)| z^{-7/3}\mathrm{Ai}(z^{2/3}b^2), & b &\ge \tfrac{\sqrt{7}-1}{6}. \end{aligned} \tag{9.4}$$

These estimates may be compared with the order estimates (5.4) obtained from less qualitative information on the functions $h_n(w)$.

Appendix. We formulate conditions on $h_0(w)$ such that the estimate of $|\varepsilon_{n|_{\mathcal{L}''}}|$ is exponentially small compared with the estimate of $|\varepsilon_{n|_{\mathcal{L}'}}|$ as $z \to \infty$ uniformly with respect to b. We take $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{L}''$, $\rho(b)$, δ, θ, $\varepsilon_{n|_{\mathcal{L}'}}$, $\varepsilon_{n|_{\mathcal{L}''}}$, and $\widetilde{h}_n$ as in §5. Define

$$\mathcal{R}(w,b,p,q,r) = r\left|w^{-q}e^{p(\frac{1}{3}w^3 - b^2 w + \frac{2}{3}b^3)}\right|.$$

We assume that $h_0(w)$ is an analytic function on a neighborhood $\Omega_0(b)$ of $\mathcal{L}''$, such that for every $w \in \mathcal{L}''$ a disc with center w and radius $\mathcal{R}$ is contained in $\Omega_0(b)$, where $r > 0$ and $p,\ q \geq 0$ do not depend on b and w. Note that, since $w \in \mathcal{L}''$, $\mathcal{R}$ may be exponentially small as $|w| \to \infty$. Furthermore, we assume that there are constants $\sigma \geq 0$ and $C_0 > 0$ such that

$$\text{(A.1)} \qquad |h_0(w)| \leq C_0 \widetilde{h}_0 |e^{-\sigma(\frac{1}{3}w^3 - b^2 w + \frac{2}{3}b^3)}| \quad \forall w \in \Omega_0(b) \cup \mathcal{L},\ b \in [0, \infty).$$

Thus we allow functions $h_0(w)$ to be exponentially large as $|w| \to \infty$.

We define recursively neighborhoods $\Omega_n(b)$ of $\mathcal{L}''$ for $n = 0, 1, 2, \ldots$. Let $\Omega_{n+1}(b)$ be those $w \in \Omega_n(b)$ such that the disc with center w and radius $2^{-(n+1)}\mathcal{R}$ is contained in $\Omega_n(b)$.

Next, let $w \in \Omega_n(b)$ and let Γ be the circle with center w and radius $2^{-n}\mathcal{R}$. The following two weak asymptotic estimates are simply proved with (4.5):

$$\text{(A.2)} \qquad \frac{1}{2\pi i} \int_\Gamma R_n(u, w, b) u^m du = \mathcal{O}(|e^{-\frac{1}{3}w^3 + b^2 w - \frac{2}{3}b^3}|),$$

$$\text{(A.3)} \qquad \sup_{u \in \Gamma} |R_n(u, w, b)| = \mathcal{O}(|e^{-((n+1)p + \frac{1}{2})(\frac{1}{3}w^3 - b^2 w + \frac{2}{3}b^3)}|)$$

as $|b| \to \infty$ uniformly with respect to $w \in \Omega_n(b)$ and $m \in \{0, 1\}$.

Now we can estimate $h_n(w)$ on $\Omega_n(b)$.

$$\begin{aligned}
h_n(w) &= \frac{1}{2\pi i} \int_\Gamma R_0(u, w, b) h_n(u) du \\
&= \frac{1}{2\pi i} \int_\Gamma R_1(u, w, b) h_{n-1}(u) du - \frac{1}{2\pi i} \int_\Gamma R_1(u, w, b)(\alpha_{n-1} + \beta_{n-1} u) du \\
&=_{(A.2)} \frac{1}{2\pi i} \int_\Gamma R_1(u, w, b) h_{n-1}(u) du + \widetilde{h}_{n-1} \mathcal{O}(|e^{-\frac{1}{3}w^3 + b^2 w - \frac{2}{3}b^3}|) \\
&\ \vdots \\
&=_{(A.2)} \frac{1}{2\pi i} \int_\Gamma R_n(u, w, b) h_0(u) du + (\widetilde{h}_{n-1} + \cdots + \widetilde{h}_0) \mathcal{O}(|e^{-\frac{1}{3}w^3 + b^2 w - \frac{2}{3}b^3}|) \\
&=_{(A.3)\ \&\ (5.3)} \widetilde{h}_0 \mathcal{O}(|e^{-(np + 1 + \sigma)(\frac{1}{3}w^3 - b^2 w + \frac{2}{3}b^3)}|).
\end{aligned}$$

Thus with (5.3) we have proved that

$$\text{(A.4)} \qquad |h_n(w)| \leq C_n \widetilde{h}_0 |e^{-(np + 1 + \sigma)(\frac{1}{3}w^3 - b^2 w + \frac{2}{3}b^3)}|$$

for all $w \in \Omega_n(b) \cup \mathcal{L}$ and $b \in [0, \infty) \cup [0, i\infty)$.

For $b \geq c > 0$ it is not difficult to prove that $\mathcal{L}'' = \{\sqrt{(1/3)y^2 + b^2} + iy \mid |y| \geq \delta' b^\theta\}$ for a certain positive δ' that does not depend on b. With the notation of §8 we have

$$\varepsilon_{n|_{\mathcal{L}''}} = (-1)^n z^{-n} \frac{e^{-\frac{2}{3}zb^3}}{2\pi i} \int_{\delta' b^\theta}^\infty e^{-z\phi(y)} \left[H_n(y) + H_n(-y)\right] dy.$$

We choose $z > np + 2 + \sigma$ and estimate $|\varepsilon_{n|_{\mathcal{L}''}}|$:

$$\begin{aligned}
|\varepsilon_{n|_{\mathcal{L}''}}| &\leq_{(A.4)} C_n' \widetilde{h}_0 e^{-\frac{2}{3}zb^3} z^{-n} \int_{\delta' b^\theta}^{\infty} e^{-(z-np-1-\sigma)\phi(y)}\, dy \\
&\leq C_n' \widetilde{h}_0 e^{-\frac{2}{3}zb^3} z^{-n} \int_{\delta' b^\theta}^{\infty} e^{-(z-np-1-\sigma)by^2}\, dy \\
&\leq C_n' \widetilde{h}_0 e^{-\frac{2}{3}zb^3} z^{-n} \frac{e^{-\delta'^2(z-np-1-\sigma)b^{2\theta+1}}}{2\delta' b^{\theta+1}(z-np-1-\sigma)} \\
&\leq C_n'' \widetilde{h}_0 \operatorname{Ai}(z^{\frac{2}{3}} b^2) z^{-n-\frac{4}{3}} e^{-\delta'^2(z-np-2-\sigma)b^{2\theta+1}}.
\end{aligned}$$

With similar estimates for $b \in [ic, i\infty)$ we have proved

$$|\varepsilon_{n|_{\mathcal{L}''}}| \leq C_n \widetilde{h}_0 \widetilde{\operatorname{Ai}}(z^{\frac{2}{3}} b^2) z^{-n-\frac{4}{3}} e^{-\delta'^2(z-np-2-\sigma)|b|^{2\theta+1}}, \tag{A.5}$$

where the constants δ' and C_n do not depend on b and z.

Remark. For the boundary case that has been handled in §6 it is not difficult to prove that $p = \sigma = 0$, and (6.10) shows that in (A.4) $\widetilde{h}_0$ can be replaced by $(b+1)^{(\theta+1/2)(\alpha+1)}|h_0(b)|$, and further calculations show that in (A.5) $\widetilde{h}_0$ can be replaced by $|h_0(b)|$.

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